INTEGRATION BY PARTS – THE D–I METHOD

This is a short cut to integration by parts and is especially useful when one has to integrate by parts several times. It is a schematic method of the traditional udv method that usually is written as \( \int udv = uv - \int vdu \) in calculus books.

We will illustrate and demonstrate by using examples. Remember that integration by parts is usually indicated when you have to integrate a product of two functions. This method works exceedingly well when one of these functions is a polynomial and the other is successively integrate. In this case it is possible to successively differentiate until the last derivative is zero.

Example 1. Find \( \int xe^x \, dx \). The format is as follows:

\[
\begin{array}{c|c|c}
\text{D} & \text{I} \\
\hline
x & e^x & + \\
1 & e^x & - \\
0 & e^x & + \\
\hline
\end{array}
\]

\( D \) means to differentiate the functions in that column. \( I \) means to integrate the functions in that column. Form the diagonal products as indicated by the arrows in the above table alternating algebraic signs as you move down the table. Finally, you integrate the product of the last entry in the \( D \) column and the last entry in the \( I \) column, with the appropriate sign, of course.

\[ \int xe^x \, dx = x\cdot e^x - 1\cdot e^x + \int 0\cdot e^x \, dx \]

which is: \( \int xe^x \, dx = xe^x - e^x + C \).

Example 2. Find \( \int x^2e^x \, dx \). The format is as follows:

\[
\begin{array}{c|c|c}
\text{D} & \text{I} \\
\hline
x^2 & e^x & + \\
2x & e^x & - \\
2 & e^x & + \\
0 & e^x & - \\
\hline
\end{array}
\]

and therefore the answer to the problem is: \( \int x^2e^x \, dx = x^2\cdot e^x - 2x\cdot e^x + 2\cdot e^x - \int 0\cdot e^x \, dx \)

or \( \int x^2e^x \, dx = x^2e^x - 2xe^x + 2e^x + C \).
Example 3. Find $\int x \sin x \, dx$.

$$
\begin{array}{c|c}
\text{D} & \text{I} \\
\hline
x & \sin x + \\
n & \cos x - \\
0 & -\sin x + \\
\end{array}
$$

and therefore the solution to the problem is: $\int x \sin x \, dx = -x \cos x + \sin x + C$.

Here we have not written the last term as $\int 0 \sin x \, dx$ but rather just as an arbitrary constant of integration $C$.

Example 4. Find $\int e^{-2x} \cos (3x) \, dx$.

Notice that here neither function is a polynomial. Since we have been picking the polynomial, x’s to powers, and differentiated until we reached zero we were able to discard the last integral of a product which was zero and merely write $C$. We can use the D–I method to work out the above integral in the following fashion:

$$
\begin{array}{c|c}
\text{D} & \text{I} \\
\hline
e^{-2x} & \cos 3x + \\
-2e^{-2x} & \sin 3x/3 - \\
4e^{-2x} & -\cos 3x/9 + \\
\end{array}
$$

Notice we stopped the process when the product of the functions in the $D$ and $I$ columns in the third row was the same as the product of the functions in the $D$ and $I$ columns in the first row, except for constants. That is we have $e^{-2x} \cos 3x$ in the first row and we have $-\frac{4}{9} e^{-2x} \cos 3x$ in the third row. They are the same except for the constant multiple $-\frac{4}{9}$.

We have: $\int e^{-2x} \cos (3x) \, dx = \frac{1}{3} e^{-2x} \sin 3x - \frac{2}{9} e^{-2x} \cos 3x - \frac{4}{9} \int e^{-2x} \cos (3x) \, dx$.

Solving for the integral we wish to evaluate, and adding an arbitrary constant of integration, we have:

$$
\int e^{-2x} \cos (3x) \, dx = \frac{3}{13} e^{-2x} \sin 3x - \frac{2}{13} e^{-2x} \cos 3x + C.
$$