A revisit of Navier-Stokes equation

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Abstract
An effort has been recently paid to derive and to better understand the Navier-Stokes (N-S) equation, and it is found that, although the N-S equation has been proven to be correct by numerous examples, some concepts and principles behind the equation may not be correct or consistent. For instance, from an analysis of the simple classic Couette flow, the requirement of the symmetric stress tensor is in fact conflicting with the solution of the Couette flow.

To solve the inconsistencies identified in this research, a reformulation of the total tensor is suggested for accommodating the fluid friction which bears a solid physics, and the new total tensor could resolve all the inconsistencies and conflicts identified. The newly defined fluid friction tensor is then used to derive N-S equation, and as expected, the same N-S equation as the original form of N-S equation for incompressible flows is obtained. For compressible flows, to achieve the same N-S equation as the original N-S equation, a slightly different assumption but yet in a very similar manner as Stokes made in 1845 is needed.

It is the author’s intention that the N-S equation under the new defined total tensor has different, but yet more physical background concepts and principles. It is hoped that the revisit of the N-S equation could shed some light to better understand the dynamic flows and lead to establish new and better approaches to solve the complicated flow problems in future.

1 Introduction
Fluid dynamics is an ancient topic and people have been trying to solve the fluid mechanics/dynamics problems since the great Greek philosophers and scientists Aristotle (384-322 BC) and Archimedes (287 – 212 BC) [1]. The first partial differential equation for fluid dynamics was much later formulated by Euler in 1752 when he considered only for inviscid fluids (that is why the fluid dynamic equation for inviscid fluids is called Euler equation). After that, by adopting the Newton’s definition of friction due to the velocity gradient and fluid viscosity, Navier and Stokes could independently include the viscous forces into the equation, thus now it is called Navier-Stokes equation. According to Anderson [2], an interesting fact is that Navier did get the correct equation in 1822, but his derivation has been greatly flawed. Stokes re-derived the fluid dynamic equation in a more delicate manner in 1845 [3]. Darrigol [4] has indicated that in the period between Navier (in 1822) to Stokes (in 1845), other three main scholars: Cauchy, Poisson and Saint-Venant, had made their contributions to establish the fluid
dynamic equation under different assumptions. Without particular reasons, the final settlement of the fluid dynamics equation might be the successful applications of Stokes’ pendulum theory in the Hagen’s Bessalian pipe flow and the Poiseuille’s capillary-tube flow (see [4]). Now it is generally accepted that the establishment of the fluid dynamics equation was finished with the work of Stokes in 1845, and the fluid dynamic equation was later named as the Navier-Stokes equation, even though Navier and Stokes published their equations independently in a gap of more than 20 years.

The Navier-Stokes (N-S) equation is the fundamental equation for governing fluid motion and dynamics, and so far numerous examples have proven the correctness of the N-S equation for fluid dynamics. However, it has been well recognised that seeking an analytical solution to the N-S equation has been proven too difficult and analytical solutions can only be obtained for some simple laminar flows, therefore, turbulence is frequently referred as the major unsolved problem of classical physics [5]. In 2000 the Navier-Stokes equation was selected to be one of seven Millennium Problems by the Clay Mathematics Institute of Cambridge, U.S. (http://www.claymath.org/millennium-problems), and a special award of $1 million is provided for the answer to each of the 7 millennium questions. In 2008 the U.S. Defense Advanced Research Projects Agency (DARPA) listed it as one of 23 DARPA Mathematical Challenges- “Mathematical Challenge Four: 21st Century Fluids”. The challenge statement is as following: although classic fluid dynamics and the Navier-Stokes Equation were extraordinarily successful in many practical problems, including the understanding of shock waves and turbulent flows, new methods (and understandings) are still needed to tackle the complex fluids such as foams, suspensions, gels, and liquid crystals (https://www.britannica.com/science/Navier-Stokes-equation).

Due to the difficulties in obtaining analytical solutions to N-S equation, especially for those real complicated turbulent flows, modern numerical methods, especially the Computation Fluid Dynamics (CFD), have been developed and advanced with the increased computer power and the advanced numerical algorithms, and nowadays the computational fluid dynamics (CFD) have been widely used to numerically solve the N-S equation, including some very complicated fluid dynamics problems, such as airplanes, air engines, ships, fire modelling, heat transfer, chemical reaction and so on. To date, CFD have been very successful for studying various fluid dynamic problems, and great achievements have been attained. Good examples are the CFD applications in development of Airbus 380 in Figure 1 [6] and of Boeing 787 in Figure 2 [7], from which we can already see how much computer modelling work can be already carried out using CFD. With the increase of computer powers, the advancement of numerical algorithms and the understanding of the complicated flows (for instance, the better turbulence models), CFD will get more and more applications and become more and more capable in solving the problems in our daily and in the complicated fluid dynamics.

Though CFD have been very successful in solving N-S equation, there are still many challenges in using CFD for complicated flows. This includes the examples mentioned above, where CFD is still limited to some specific problems and the explorations are still on going to use more CFD in the design of the whole aircrafts and the off-design situations. In addition, the needs to use CFD to solve the multi-physical problems [6-10] and seek global optima [7] are the current challenges.

The current approaches in CFD include the most computational demanding approach: direct numerical simulation (DNS, [11, 12]); relatively less (compared to DNS) but still very computational demanding method: large-eddy simulation (LES, [13-17]) and the most practical method: the Reynolds-average N-S (RANS, [5, 18-30]). The RANS method resolves the mean flow numerically, and models the turbulent
flow via turbulence models, and it has the advantages of the moderate requirements in grid and temporal stepping (thus the generally accepted computational burden), and of a good numerical convergence. The method has been proven to be very useful in many practical flows, including the examples above. However, it is also generally accepted the RANS method may not be very reliable in predicting the complicated flows since different turbulence models may lead to different flow simulations. In the past decades, many researchers have tried to tune/modify/extend the turbulence models, but mostly for their specific problems, other than for more general turbulence models. As a result of such difficulties, so far there is hardly any universal turbulent model for different flows, and we may even have difficulties in many general flow problems, such as flow transition from laminar to turbulent; the flows with adverse pressure gradients; and flow separations and re-attachments. The author thinks that it may be a good idea to go back to the very fundamentals in fluid dynamics, and the enhancement and better understanding of some concepts and principles behind the N-S equation may be needed, and believes that a better physical understanding and significance to the fundamentals of fluid dynamic equation could pave a path to construct better turbulence models for and thus the better solutions to the complicated flows. This is the main intention of the research.
This research examines the very fundamentals of the N-S equation, for instance, the symmetric viscous stress tensor in fluids introduced by Stoke in 1845 (named as the Stokes’ symmetric stress tensor which originated from the Cauchy’s symmetric stress tensor for elastic materials). This Stokes’ symmetric stress tensor was so well accepted in the community that it has been taken as a law that all fluids must follow, in a similar manner as the Newton’s laws of motion as the universal law of motion. However, as stated in [31], the symmetry in the stress tensor is violated in an electric field on polarized fluid molecules, in which antisymmetric stresses must be included in such circumstances. A similar contradiction has been reported in [32]. The author is wondered why the Stokes’ symmetric stress tensor is violated in some special flows?

To find an answer to the question above, the author has recently made an effort to derive and better understand the N-S equation. It is found that, although the N-S equation is proven to be correct for governing all various flows, including the compressible flows with the Stokes’ assumption [31] (page 128), some concepts and principles behind the equation may not be correct or consistent. For instance, an analysis to the classic Couette flow has shown that the requirement of the symmetric stress tensor is conflicting with the solution of the Couette flow when a real physics is applied in the analyses (more details on these conflicts/inconsistencies can be found in Section 4).

To understand the problems mentioned above, the forces acting on the fluids are re-examined which could show the inconsistencies for the Stokes’ symmetric stress tensor and a reformulation has been made to the total tensor for accommodating the total surface force of more physical significance. Using the newly defined friction tensor, an N-S equation exactly same as the original form of N-S equation is obtained for incompressible flows. It is also possible to obtain a same N-S equation for compressible flows using a slightly different assumption as Stokes made. Therefore, under the newly defined total tensor, the N-S equation is maintained same, however, the concepts and principles behind the equation are different but of more physical significance.

To present the research work, following arrangement is made for the rest of the paper. Section 2 presents the simple introduction and derivation of the original form of N-S equation, with some discussions for the equation; in Section 3 viscous stress tensor and surface forces are illustrated, providing better understandings and physical significance to the viscous stress tensor and its components; Section 4 lays out 3 inconsistencies behind the N-S equation, all based on the solid physical understanding to the force analyses for the fluids; in Section 5, more details of fluid motions and forces are presented and the asymmetric friction tensor is proposed; in Section 6 Navier-Stokes equation is derived using the new total tensor for both incompressible and compressible flows; and in Section 7, some more discussions are made for the newly defined asymmetric stress tensor. In the last section, conclusions are provided for the research work.

2 The original form of N-S equation

In this section, the original N-S equation is first derived briefly, to provide a base for further analyses to what problems may have behind the N-S equation.

2.1 Conservation of momentum

The N-S equation can be easily derived from the transport theorem and the conservation of momentum (note: the fluid dynamic equation can also be derived in different ways, see [33, 34]).
Based on the Newton second law of motion, the momentum of the fluid dynamics can be expressed as,

\[
\frac{D}{Dt} \iiint_V \rho \mathbf{N} dV = \iiint_V \mathbf{f} dV + \iiint_S \mathbf{\tilde{T}} \cdot \mathbf{n} dS
\]  

(1)

In the equation, \( \mathbf{V} \) and \( V \) represent different variables: the former is the fluid velocity (vector) and the latter the fluid volume (scalar); \( \mathbf{\tilde{T}} \) is the total tensor (the double-sided arrow means a tensor) for surface stress (a force per unit surface area); \( \mathbf{f} \) the body force (force per unit volume). For many applications, the simplest body force is the gravitational force of the fluid, that is, \( \mathbf{f} = \rho \mathbf{g} \mathbf{k} \), with \( g \) being the gravitational acceleration, \( \mathbf{k} \) the unit vector along \( z \)-axis.

By invoking the Gauss divergence theorem (see Appendix A2) on the last term in the right-hand-side of Eq. (1), the momentum conservation can be expressed as,

\[
\frac{D}{Dt} \iiint_V \rho \mathbf{N} dV = \iiint_V (\mathbf{f} + \nabla \cdot \mathbf{\tilde{T}}) dV
\]  

(2)

invoking the transport theorem, the conservation of momentum (see Appendix A3) leads to,

\[
\frac{\partial (\rho \mathbf{N})}{\partial t} + \nabla \cdot (\rho \mathbf{V V}) = \mathbf{f} + \nabla \cdot \mathbf{\tilde{T}}
\]  

(3)

Using the Einstein summation convention, the conservation of momentum can be simply expressed as

\[
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = f_i + \frac{\partial T_{ij}}{\partial x_j}
\]  

(4)

where \( f_i \) is the body force component and hereafter the subscripts \( i, j = 1, 2, 3 \) mean the components along \( x \)-, \( y \)- and \( z \)-axes respectively.

2.2 Total and stress tensors in fluids

Following the general convention, the total stress tensor has a general form as,

\[
\begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
\]  

(5)

As such, the surface force (per unit surface area) acting on a surface is calculated as

\[
\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \mathbf{\tilde{T}} \cdot \mathbf{n} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} T_{11}n_1 + T_{12}n_2 + T_{13}n_3 \\
T_{21}n_1 + T_{22}n_2 + T_{23}n_3 \\
T_{31}n_1 + T_{32}n_2 + T_{33}n_3 \end{bmatrix}
\]  

(6)

where \( F_1, F_2, \) and \( F_3 \) are three components along \( x \)-, \( y \)- and \( z \)-axes; \( \mathbf{n} \) is the normal vector of the surface, and \( n_1, n_2 \) and \( n_3 \) are the components of the normal vector along \( x_1 \)-, \( x_2 \)- and \( x_3 \)-axes, respectively (see Figure 3).
Figure 3 A stress force (vector) on a small surface, $\Delta S$

In details, the definition of the total tensor was proposed by Stokes [3] as

$$\vec{T} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} + \begin{bmatrix} \lambda \nabla \cdot \mathbf{V} & 0 & 0 \\ 0 & \lambda \nabla \cdot \mathbf{V} & 0 \\ 0 & 0 & \lambda \nabla \cdot \mathbf{V} \end{bmatrix}$$  \hspace{1cm} (7)

where $\lambda$ is so called the second viscosity coefficient (see [33]), which is different from the physical viscosity coefficient $\mu$ of the fluid.

Based on the conservation of angular momentum (from the Cauchy’s second law of motion), the viscous stress tensor must be symmetric, and the currently accepted stress components are given as

$$\begin{cases}
\tau_{11} = 2\mu \frac{\partial u_1}{\partial x_1}; & \tau_{22} = 2\mu \frac{\partial u_2}{\partial x_2}; & \tau_{33} = 2\mu \frac{\partial u_3}{\partial x_3} \\
\tau_{12} = \tau_{21} = \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\
\tau_{13} = \tau_{31} = \mu \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \\
\tau_{23} = \tau_{32} = \mu \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right)
\end{cases}$$  \hspace{1cm} (8)

The first term in the right-hand-side of Eq. (7) is pressure tensor, and the diagonal tensor means the pressure could produce only normal forces on the surfaces (the negative sign means the pressure force pointing into the body); the second term is the stress tensor due to the fluid viscosity, which is very similar to pressure and has the same unit as the pressure. The difference between the pressure force and the stresses is that the pressure is a scalar at one point in the fluid, but the stress is a tensor, of which the components have orientations when act on a surface. For example, the stress component, $\tau_{21}$, denotes a stress acting in the $y$-direction on a surface of constant $x$; and $\tau_{11}$, denotes a stress acting in the $x$-direction on the surface of constant $y$. Similarly, $\tau_{11}$ denotes a normal stress acting on a surface of constant $x$ (see [35, 36]).

The third term is a term of the fluid volume dilatation contributing to the total stress tensor, and Stokes [3] added for accommodating the fluid compressibility. But it is still a controversial term [37, 38].
Obviously, this term disappears for incompressible flows. It is reported that it is very important in the analysis of the shock wave structure where the pressure and temperature could change dramatically in short time and short distances \([33]\). Stokes \([3]\) also proposed a relation between the physical and the secondary viscosity coefficients as

\[
\lambda = -\frac{2}{3} \mu
\]

(9)

It should be noted that this relation is still controversial based on the textbook ([39], page 114).

### 2.3 Navier-Stokes equation

Based on the principles and assumptions above, the full Navier-Stokes equation can be derived as

\[
\begin{align*}
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_j u_i)}{\partial x_j} + \frac{\partial (\rho u_i u_j)}{\partial x_j} &= f_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \frac{1}{3} \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
\frac{\partial (\rho u_j u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} + \frac{\partial (\rho u_i u_j)}{\partial x_j} &= f_j - \frac{\partial p}{\partial x_j} + \mu \nabla^2 u_j + \frac{1}{3} \mu \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\end{align*}
\]

(10)

This is the widely accepted N-S equation for both compressible and incompressible flows.

In a vector form, the Navier-Stokes equation is expressed as

\[
\frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V} \mathbf{V}) = \mathbf{f} - \nabla p + \mu \nabla^2 \mathbf{V} + \frac{1}{3} \mu \nabla \left( \nabla \cdot \mathbf{V} \right)
\]

(11)

or in the Einstein summation convention,

\[
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_j u_i)}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \frac{1}{3} \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

(12)

### 3 Viscous stress tensor and surface forces

Here the viscous stress tensor means the part of the stress tensor due to the fluid viscosity, written as

\[
\mathbf{\tau} = \begin{bmatrix}
\tau_{11} & \tau_{12} & \tau_{13} \\
\tau_{21} & \tau_{22} & \tau_{23} \\
\tau_{31} & \tau_{32} & \tau_{33}
\end{bmatrix}
\]

(13)

From the textbooks and research papers, it is frequently seen that the stress tensor components are shown in Figure 4 (note: on surface \(S_2\) the normal points opposite to \(x_2\)-direction). However, to better understand the figure, some delicate explanations must be given to the illustration. Taking the small cube (enlarged for illustration) and considering the special orientations of the chosen plates, \(S_1, S_2\) and \(S_3\), using Eq. (6), we can easily show that there are only three simple surface force components on each of those specific planes, see Eq. (14) and Figure 5(a). Here the normal vectors for these specific planes are still indicated by \(n_1, n_2\) and \(n_3\) for a clarification.
For the specific faces $S_1$, $S_2$ and $S_3$, their normal vectors are actually $n_1 = 1$; $n_2 = -1$; $n_3 = 1$, therefore,

\[
\begin{align*}
F_{x1} &= \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \tau_{11}n_1 \\ \tau_{21}n_2 \\ \tau_{31}n_3 \end{bmatrix} = \begin{bmatrix} \tau_{11} \\ -\tau_{21} \\ \tau_{31} \end{bmatrix} \\
F_{x2} &= \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \tau_{11}n_1 \\ \tau_{22}n_2 \\ \tau_{32}n_3 \end{bmatrix} = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{32} \end{bmatrix} \\
F_{x3} &= \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \tau_{13}n_3 \\ \tau_{23}n_3 \\ \tau_{33}n_3 \end{bmatrix} = \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{bmatrix}
\end{align*}
\]

The transformed result is shown in Figure 5(b), which is actually the same as those in Figure 4. This explains why the surface forces can be specified using the stress tensor components (on $S_2$, the negative signs mean the opposite directions of the force components).
4 Inconsistencies on shear stress for N-S equation

4.1 Fluid definition and shear stress in fluids

Let us go back to the very fundamental concept for fluid: what is fluid? Here a stricter technical or physical definition of fluid is referred: based on Britannica Encyclopaedia (https://www.britannica.com/science/fluid-physics), Fluid, including any liquid or gas or generally any other material, can not sustain a tangential/shear force when at rest and could undergo a continuous change in shape under such a stress. In other word, under a shear stress (regardless how small it could be), a continuous and irrecoverable change of position of the material forms a flow, which is a very basic property of fluids. In contrast, the shear stresses can be maintained within a deformed elastic solid, and the deformed solid could spring back to its original shape when the stresses are removed.

While in Bertin and Smith [33], it states that the technical distinction between a fluid and a solid lies with their reaction to an applied shear/tangential stress acting to them: a solid can resist a shear stress by a static deflection; a fluid can not. Any shear stress applied to a fluid, no matter how small, will result in a flow of that fluid, a continuous deformation of the shape. A very similar statement can be found in White [40].

In both definitions, a significant distinction between a fluid and a solid is that a fluid can not resist a shear stress without a permanent deformation (i.e., flowing) regardless how small of the shear stress could be, while a solid could spring back to original form after the removal of the shear stress acting on the solid (if the solid deformation is within the elastic deformation). In a word, fluids can only have shear stress components when it flows. This is a very important statement which will be used for discussing the inconsistencies on the shear stress of fluids.

4.2 Inconsistency 1: symmetric shear stress in fluids

The Cauchy’s second law of motion is based on the assumption of the conservation of angular momentum. This law was proposed by Cauchy for elastic materials that the stress tensor must be symmetric in the equilibrium state due to the force balance.
The Cauchy’s concept of the symmetric stress tensor was adopted for fluid dynamics by Stokes in 1845 [3]. Analogous to the Newton’s formula for fluid friction, Stokes formulated the symmetric stress tensor, i.e., the viscous stress tensor in fluid dynamics, and the tensor components given in Eq. (8). To date, this is well accepted definition, and hardly there are any doubts for this definition since 1845, even though some research results have shown that the viscous stress tensor may be asymmetric for some special flows, such as compressible flows as seen in the cases of rarefied gases, shock waves and gaseous flows through micro fluidic channels [32, 37, 41, 42], but these research results have been only taken as the exceptional cases in the fluid dynamics community. For instance, Wilcox [24] has especially stated that the symmetric stress tensor is for simple viscous fluids, but not for some anisotropic liquids (see page 39). However, no reason is given for the statement.

Another example for the inconsistent symmetrical stress tensor is given in the textbook of Kundu etc ([31], page 126), with a statement as: the stress tensor symmetry is violated in the electric field on polarized fluid molecules, where antisymmetric stresses must be included in the analysis.

A question would be if the Cauchy’s law of angular moment conservation is a universal law for fluids, which requires the fluid viscous shear stress must be symmetric. Then why there are so many exceptions? After all, a universal law must be satisfied for all cases as we do not see any exceptions for Newton’s laws of motion. Obviously, there is an inconsistency in some practical examples when the fluid viscous stress tensor is required to be symmetric. Hence the inconsistency is termed as Inconsistency 1 here.

4.3 Inconsistency 2

According to the friction definition of a Newtonian fluid, if a fluid velocity gradient is between adjacent fluid particles, a fluid friction occurs due to the fluid viscosity, as

\[ \tau_{12} (\tau_{xy}) = \mu \frac{\partial u}{\partial y} \] (16)

This is actually a formula for defining the fluid viscosity coefficient, and an illustration can be seen in Figure 6(a).

(a) 

(b)

\[ \tau_{11} = \mu \frac{u_2 - u_1}{\Delta x} = \mu \frac{\partial u}{\partial x} \]

Figure 6 Shear stress due to the velocity gradient

From this example in Figure 6(a), it can be seen obviously that the shear stress or friction given by Eq.(16), the shear stress tensor component, \( \tau_{12} \), has a direction along with the velocity increment, \( \Delta u \), that is, \( \tau_{12} = \mu \frac{\partial u}{\partial y} \) has a direction on x-axis. Similarly, in Figure 6(b), the normal stress, \( \tau_{11} \), has a direction with the velocity increment, \( \Delta u \), hence the normal stress \( \tau_{11} = \mu \frac{\partial u}{\partial x} \) has a direction on x-axis.
Hence, $\tau_{11}$ and $\tau_{12}$ both have a direction in $x$-axis, but acting on different surfaces of constant $x$ and constant $y$, respectively, see Figure 5.

It will be seen later in Eq. (27) that the fluid friction direction has always the same direction with the velocity increment, regardless of whether the velocity increment is on $x$-, $y$- or $z$-directions.

Next the symmetric shear stress is examined. Taking $\tau_{12}$ as an example, it is a stress acting in the $x$-direction (subscript ‘1’) upon a surface of constant $y$ (subscript ‘2’) (see [33, 35] and Figure 5(b)). Based on the Stokes’ symmetric stress tensor, the tensor component is defined as,

$$\tau_{12} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{\partial u}{\partial y} + \mu \frac{\partial v}{\partial x}$$ (17)

This tensor component consists of two terms in the right-hand-side in Eq.(17). In physics as shown above, the first shear stress term has a direction along $x$-axis due to the increment $\Delta u$ while the second shear stress term has a direction along $y$-axis due to $\Delta v$. These two orthogonal stress terms (along $x$- and $y$-axes respectively) together make a single stress tensor component $\tau_{12}$. In addition, following the definition of stress tensor definition in [35], the tensor component, $\tau_{12}$, is acting in $x$-axis on a plane of constant $y$, while its symmetric pair $\tau_{21}$ (of a same definition) is acting along the $y$-direction on a plane of constant $x$. From the physical standpoint, the symmetric shear stress component is ambiguously defined, which has no clear physical significance.

Therefore, a question will be: how a shear stress acting in one direction could physically hold two terms of different orientations of frictions? Here this conflict is termed as **Inconsistency 2**.

### 4.4 Inconsistency 3

Here we examine the classic Couette flow. The Couette flow is confined between two large parallel plates: the lower plate is fixed and the upper plate moves at a constant speed ($u_0$). The speed of the upper plate is taken to be relatively small, so that the flow between the plates can be regarded as a laminar flow, see Figure 7(a).

![Couette flow: Fluid flow between two plates](image)

The dynamic problem can be simplified mathematically as a 2D flow as shown in Figure 7(b). From the plot, we have: $u = u(y)$, and $v = w = 0$. As such, the N-S equation (zero pressure gradient here) is degraded to,

$$\frac{\partial^2 u}{\partial y^2} = 0$$ (18)

Applying the boundary conditions: $u = 0$ at $y = 0$; $u = u_0$ at $y = h$, the solution of the Eq. (18) is
\[ u = \frac{y}{h} u_0 \]  

(19)

This is the analytical solution of the Couette flow, and the flow velocity distribution is shown in Figure 7(b).

Based on the definition of the symmetric stress tensor components in Eq. (8), we have

\[ \tau_{12} = \mu \frac{\partial u}{\partial y} + \mu \frac{\partial v}{\partial x} = \frac{\mu u_0}{h} \]

(20)

Following the definition of the orientation of the stress tensor component (see Figure 5), it is a stress in \( x \)-axis, which is caused due to the different velocities of \( u \) in \( y \)-direction.

From the symmetrical stress tensor, it requires

\[ \tau_{21}(= \tau_{12}) = \mu \frac{u_0}{h} \]

(21)

It must be noted that this is a stress in \( y \)-direction.

As shown in Figure 8, for a small rectangular fluid element, on its four sides, there are 4 stresses: \( \tau_{12} \) & \( \tau_{21} \), \( \tau_{12}' \) & \( \tau_{21}' \) making two pairs of stresses due to the symmetric stress tensor. For the Couette flow, the horizontal viscous stresses \( \tau_{12} \) and \( \tau_{12}' \) on the upper and lower sides can be easily understood, and they have an obvious physical foundation, since the stress in \( x \)-direction can cause a flow of the fluid (a constant deformation) in the same direction. However, from the requirement of the symmetric stress tensor the vertical viscous stresses \( \tau_{21} \) and \( \tau_{21}' \) must exist as shown in Figure 8. The question here is what causes the vertical stresses in the Couette flow? Or if there exist vertical stresses \( \tau_{21} \) and \( \tau_{21}' \), why they do not cause a flow along \( y \)-direction?

As it is already shown in the fluid definition, a shear stress in a fluid can only exist when it flows. But for the Couette flow, there is only a flow in \( x \)-direction, but no flow in \( y \)-direction, then how the vertical shear stresses are caused?

The similar conflict can be seen from the solution of the classic flow in a horizontal pipe (see examples in [34]).

The shear stress acting on a fluid causes a flow of the fluid, and as a result of the flow, the flow induced fluid friction could balance the shear stress. This is very different from the solids in which a force balance must be required on the conservation of the angular moment, that is, the symmetric stress tensor in solids.

Therefore, this is a conflict, termed as **Inconsistency 3** in this research.
5 New understanding of fluid motion and force

5.1 Relative velocity between two points in fluid

At a certain time $t$, the fluid point $A$ is taken as a reference point, given by a position vector $\mathbf{r}_A=(x_0, y_0, z_0)^T$, and the corresponding velocity components as $(u_0, v_0, w_0)^T$. At the same time, one of its neighbouring point $B$ is positioned by a vector, $\mathbf{r}_B=(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)^T$. So the vector from $A$ to $B$ is calculated as $\mathbf{dr}=(\Delta x, \Delta y, \Delta z)^T$.

The corresponding velocity components at $B$ can be simply put as

$$
\begin{align*}
    u_1 &= u_0 + \Delta u \\
    v_1 &= v_0 + \Delta v \\
    w_1 &= w_0 + \Delta w
\end{align*}
$$

(22)

The fluid friction (from the friction definition of a Newtonian fluid) is caused due to the velocity difference between the two neighbouring points, $A$ and $B$, i.e., the relative velocity components ($B$ to $A$) can be calculated as
\[
\begin{bmatrix}
\Delta u \\
\Delta v \\
\Delta w
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z \\
\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \frac{\partial v}{\partial z} \Delta z \\
\frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \frac{\partial w}{\partial z} \Delta z
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{bmatrix}
\] (23)

From Eq. (23), it can be seen that the directions of fluid friction is always in the same direction with the increment of velocity component, that is, \( \Delta u \) in x-direction, \( \Delta v \) in y-direction and \( \Delta w \) in z-direction.

### 5.2 Fluid friction tensor

The relative velocity components between the joint fluid points are the requisite condition for inducing viscous forces (frictions) in fluid. Based on this physical understanding and the definition of fluid friction (note: friction is used here to avoid using the word ‘stress’), it is reasonable to define a friction tensor \( \widetilde{\sigma} \) as

\[
\widetilde{\sigma} = \begin{bmatrix}
\mu \frac{\partial u}{\partial x} & \mu \frac{\partial u}{\partial y} & \mu \frac{\partial u}{\partial z} \\
\mu \frac{\partial v}{\partial x} & \mu \frac{\partial v}{\partial y} & \mu \frac{\partial v}{\partial z} \\
\mu \frac{\partial w}{\partial x} & \mu \frac{\partial w}{\partial y} & \mu \frac{\partial w}{\partial z}
\end{bmatrix} = \mu \widetilde{a}
\]

(24)

with the friction tensor components being given as

\[
\begin{align*}
s_{11} &= \mu \frac{\partial u}{\partial x}; & s_{12} &= \mu \frac{\partial u}{\partial y}; & s_{13} &= \mu \frac{\partial u}{\partial z} \\
s_{21} &= \mu \frac{\partial v}{\partial x}; & s_{22} &= \mu \frac{\partial v}{\partial y}; & s_{23} &= \mu \frac{\partial v}{\partial z} \\
s_{31} &= \mu \frac{\partial w}{\partial x}; & s_{32} &= \mu \frac{\partial w}{\partial y}; & s_{33} &= \mu \frac{\partial w}{\partial z}
\end{align*}
\]

(25)

Obviously, the friction tensor is an asymmetric tensor, which is different from the symmetric stress tensor, shown in Eq. (8).

And the fluid velocity gradient tensor (an asymmetric tensor) defined as

\[
\widetilde{a} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{bmatrix}
= \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix}
\]

(26)

The corresponding friction force on a unit surface is given as
Now we come back to the Couette flow, the friction tensor component \( \tau'_{12} \) give the friction along x-axis as

\[
\sigma_{12} = \mu \frac{\partial u}{\partial y} = \mu \frac{u_0}{h}
\]  

(28)

but the corresponding vertical shear stress component is

\[
\sigma_{21} = \mu \frac{\partial v}{\partial x} = 0
\]  

(29)

Obviously, there is no friction on y-direction. Therefore in this way the physical conflict of the stresses in the Couette flow is resolved.

5.3 Analysis of velocity gradient tensor

The velocity gradient tensor can be rewritten as

\[
\tilde{a} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{bmatrix} + \begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
\]  

(30)

with the symmetric part as following,

\[
\begin{align*}
S_{11} &= \frac{\partial u}{\partial x}; & S_{22} &= \frac{\partial v}{\partial y}; & S_{33} &= \frac{\partial w}{\partial z}; \\
S_{12} &= S_{21} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right); \\
S_{13} &= S_{31} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right); \\
S_{23} &= S_{32} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)
\end{align*}
\]  

(31)

and the anti-symmetric part with the tensor components defined as

\[
\omega_1 = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right); \quad \omega_2 = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right); \quad \omega_3 = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
\]  

(32)

here \( \omega_1, \omega_2, \omega_3 \) are the components of the angular velocity vector or vorticity vector, \( \omega \), following the definition given in [34],

\[
\omega = \frac{1}{2} \nabla \times \mathbf{V}
\]  

(33)
From Eq. (30), it can be seen that the velocity gradient tensor can be written into two parts: the symmetric part and the anti-symmetric part. The symmetric strain-rate has been adopted in Stokes stress tensor for deriving the N-S equation.

The basic difference between the symmetric stress tensor and the asymmetric friction tensor is the last pure rotation term in Eq. (30). If we consider a special case of an irrotational flow, we have

$$\nabla \times \mathbf{V} = 0$$  \hspace{1cm} (34)

Under such a flow condition, the last term in Eq. (30) disappears. Therefore, these two tensors: the symmetric stress tensor and the asymmetric friction tensor; are exactly same.

For the Couette flow, we can show that it is rotational, since

$$\omega = -\frac{1}{2} \frac{u}{h} \neq 0$$  \hspace{1cm} (35)

meaning,

$$\nabla \times \mathbf{V} \neq 0$$  \hspace{1cm} (36)

From the practical point of view, the rotational flow condition holds in most flows within the viscous boundary layers, in which the perfect pairs of the viscous stress tensor components do not exist.

6 New concepts for Navier-Stokes equation

6.1 Surface force for fluid dynamics

For fluid dynamics, the forces acting on the fluid include both the body and surface forces, with the surface force being calculated based on the total tensor, \( T' \), as shown in Eq. (1). However, based on the discussions above, a change in the total tensor is proposed to make it more physical.

Now the new total tensor \( T' \) is defined as

\[
\begin{bmatrix}
  -p & 0 & 0 \\
  0 & -p & 0 \\
  0 & 0 & -p
\end{bmatrix}
\begin{bmatrix}
  \lambda \nabla \cdot \mathbf{V} & 0 & 0 \\
  0 & \lambda \nabla \cdot \mathbf{V} & 0 \\
  0 & 0 & \lambda \nabla \cdot \mathbf{V}
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial u}{\partial x} \\
  \frac{\partial u}{\partial y} \\
  \frac{\partial u}{\partial z}
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial v}{\partial x} \\
  \frac{\partial v}{\partial y} \\
  \frac{\partial v}{\partial z}
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial w}{\partial x} \\
  \frac{\partial w}{\partial y} \\
  \frac{\partial w}{\partial z}
\end{bmatrix}
\]

In this new definition of the total tensor, the symmetric stress tensor is replaced by the friction tensor, such a new definition (the friction from the stress tensor) could avoid the conflicts behind the original N-S equation, including:

- In fluids, shear stresses may only exist when the fluids are flowing. This is very different from the solids. Hence, the requirement of the symmetric stress tensor developed for solids may not be correct for fluids, because we have already had exceptions (see [24, 32, 37, 41, 42]). In [31] (see page 126), it states that the stress tensor symmetry is violated in the electric field on polarized fluid molecules, where antisymmetric stresses must be included in the analysis. In fact, this problem can be solved using the asymmetric friction tensor proposed in the research, Eq. (30), because both the symmetric and anti-symmetric stresses have been automatically included. As such, Inconsistency 1 is resolved.
- All components of the fluid friction tensor have a correctly-defined physical property, including the consistent and unique orientations of the frictions, hence Inconsistency 2 is resolved.
- Using the new friction tensor in Eq. (24), rather than the symmetric viscous stress tensor defined in Eq. (8), Inconsistency 3 for the Couette flow is removed, shown by Eqs. (28) and (29).

6.2 Incompressible fluids

For an incompressible fluid, its density is constant. Together with $\nabla \cdot \mathbf{V} = 0$, Eq. (3) can be rewritten as

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{\rho} \left( \mathbf{f}_g + \nabla \cdot \mathbf{T}' \right)$$

with

$$\nabla \cdot \mathbf{T}' = \left( \begin{array}{c}
- \frac{\partial p}{\partial x} + \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\
+ \frac{\partial p}{\partial y} + \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \\
+ \frac{\partial p}{\partial z} + \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}
\end{array} \right) \mathbf{i}$$


Using the new friction tensor of Eq. (25),

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \mu \frac{\partial^2 u_1}{\partial x_1^2} + \mu \frac{\partial^2 u_1}{\partial x_2^2} + \mu \frac{\partial^2 u_1}{\partial x_3^2} = \mu \nabla^2 u_1$$

Similarly, we have

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \mu \nabla^2 u_2$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \mu \nabla^2 u_3$$

Putting these together, we have the Navier-Stokes equation for the incompressible fluid as

$$\begin{align*}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} &= \frac{1}{\rho} f_1 - \frac{1}{\rho} \frac{\partial p}{\partial x_1} + \nu \nabla^2 u_1 \\
\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3} &= \frac{1}{\rho} f_2 - \frac{1}{\rho} \frac{\partial p}{\partial x_2} + \nu \nabla^2 u_2 \\
\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3} &= \frac{1}{\rho} f_3 - \frac{1}{\rho} \frac{\partial p}{\partial x_3} + \nu \nabla^2 u_3
\end{align*}$$

In a vector form, it is

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{1}{\rho} \left( \mathbf{f}_g - \nabla p + \nu \nabla^2 \mathbf{V} \right)$$

This is exactly same N-S equation for incompressible flows as that of original form of N-S equation.

This means that the N-S equation is exactly same as the original form as Stokes established in 1845, however, some concepts and principles are different and of more physics.
6.3 Compressible fluids

For compressible flows, the Navier-Stokes equation can be derived as

$$\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = \mathbf{f}_B - \nabla p + \mu \nabla^2 \mathbf{V} + \lambda \nabla (\nabla \cdot \mathbf{V})$$  \hspace{1cm} (44)

To make the equation exactly same as the original Navier-Stokes equation for compressible flows as Eq. (12), it is assumed as

$$\lambda = \frac{1}{3} \mu $$ \hspace{1cm} (45)

which is slightly different from the proposal that Stokes made in 1845, see Eq. (9). This difference is caused because of the different symmetric stress tensor and the asymmetric friction tensor.

The choice of Eq. (45) seems reasonable since the rate of volume dilatation appears 3 times in the Navier-Stokes equation (in each component in the Cartesian coordinate system), so the coefficient of 1/3 is taken. However, more work must be done on how the second viscosity coefficient can be decided.

7 Discussions

7.1 A shear stress applied on a solid element and a fluid element

Here we will see what will happen if a shear stress is applied on a solid element and a fluid element, and will explain why the asymmetrical stress tensor will not cause a fluid spinning in the fluid.

If the element is an elastic material (e.g., a solid), under the shear stress $\tau_{xz}$, there will be a deformation of the element. However, due to the elasticity, the element will produce a shear stress $\tau_{zx}$ to balance the applied shear stress and avoid an infinite rotation of the element as seen in Figure 10. Also, $\tau_{zx} = \tau_{xz}$ (this is required for solids by the Cauchy symmetric stress tensor).

![Figure 10 A shear stress $\tau_{xz}$ is applied on a solid element](image)

If the element is a fluid, the applied shear stress, $\tau_{xz}$, will simply cause a flow of the element (see based on the fluid definition). Due to the flow motion, a fluid friction is induced which would balance the applied shear stress. Therefore, this shows that there will be no shear stress pair to balance the applied shear stress in fluids, but a fluid friction to balance the applied shear stress (see Figure 11), calculated as,

$$\tau_{xz} = \mu \frac{\partial u}{\partial z}$$ \hspace{1cm} (46)
7.2 A proof for no vertical shear stress in the Couette flow

It may be argued that there may be a non-zero vertical shear stress $\tau_{yx}$ in the Couette flow, but the upper plate stops the vertical flow. Imagine if we make an open on the upper plate, there will be a flow coming out from the opening because of the vertical shear stress (see Figure 12). Is this true?

However, the answer is NO. Because in reality, whether there will be a flow flowing out from the opening totally depending on the pressure of the fluid between the plates. If the flow speed is large enough, there may be a suction of flow in the opening.

Another proof is the experiment setup similar to the Bernoulli’s principle. At the bottom of the water tank, two different size pipes are connected. Different sizes of the pipes would allow different flow velocities, as $V_1$ and $V_2$. Based on the Bernoulli’s principle, we could have different water heights in the small tubes, as $h_1$ and $h_2$.

If the non-zero vertical stress, $\tau_{yx}$, exists, it surely induce flows in the vertical small tubes (based on the fluid definition). Therefore, there will be outflows in the small tubes regardless of the height of the tubes because of the non-zero vertical shear stress. Obviously, this is not true. The heights of the fluid in the small tubes, $h_1$ and $h_2$, totally depend on the pressures of the flow in the pipes.
8 Conclusions

The research has examined some fundamental issues behind the Navier-Stokes equation, and it is identified that there exist some conflicts on the concepts and principles in deriving the N-S equation. To solve these conflicts, the forces acting on the fluids have been re-analysed based on a solid physics. By considering the fundamental properties of fluids, it can simply show that the symmetric stress tensor is violated in the classic Couette flow.

To resolve these conflicts behind N-S equation, it is proposed that the fluid friction is regarded as the special surface force, not a part of the Stokes’ symmetric stress. Under this new concept, the friction tensor is independent of the stress tensor, and more importantly, the friction tensor is not required to be symmetric if the real physics of the fluid frictions are referred to.

From the study and the understanding of the concepts and principles behind N-S equation in this research work, the following conclusions can be drawn:

- Identify the conflicts of the concepts and principles in deriving the N-S equation.
- Illustrate a conflict on the requirement of the symmetric stress tensor using the well-known Couette flow.
- Reformulate the total tensor by replacing the symmetric stress tensor with the fluid friction tensor, and the friction tensor is more based on the real physics of the friction acting on the fluids.
- The formulation of new friction tensor could resolve all identified inconsistencies, including the shear stress in the Couette flow and the special problem as in the electric field on polarized fluid molecules, since the anti-symmetric tensor is automatically included in the new total tensor formulation.
- The exactly same N-S equation as the original N-S equation can be derived using the newly defined total tensor for the incompressible flows. However, to obtain the same N-S equation for compressible flows, a slight different yet very similar assumption from the Stokes’ assumption should be taken.

The new friction tensor does lead to the exactly same N-S equation, hence it is hoped that this revisit of N-S equation with the newly proposed friction tensor could provide a better physics in understanding the N-S equation. With the better physics as proposed in this research, it may pave a way to understand the fluid dynamics better, especially for those complicated flows. For instance, currently, solving the N-S equation uses turbulence models in most practical applications, and the turbulence models have
been frequently constructed using the symmetric stress tensor following the requirement of the symmetric strain-rate tensor, see [21, 43]. Under the new physical significance presented in the research work, the asymmetric friction tensor could allow different views on the complex flows and it may shed some light for reconstructing different turbulence models so to help to solve the fluid dynamic problems better.

**Appendix: Useful mathematical tools**

**A1. Material derivative**

In the Euler expression, a given physical function, \( f \), in the fluid motion can be expressed as the function of the independent variables: the coordinates \((x, y, z)\) and time, \( t \), as

\[
f = f(x, y, z; t)
\]  

(A.1)

Note: the arbitrary function can be either a scalar, such as the fluid density, \( \rho \); pressure, \( p \), etc) or a vector, such as, the fluid velocity, \( \mathbf{V} \), or the force, \( \mathbf{F} \) and so on (hereafter the bold letters denote vectors).

The material derivative of the function can be easily obtained using the chain rule of derivative as

\[
\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}
\]  

(A.2)

In a more compact manner, it is

\[
\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{V} \cdot \nabla) f
\]  

(A.3)

The flow acceleration is the material derivative of the flow velocity vector with regard to time and it can be derived in similar manner as above, so

\[
\frac{DV}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V}
\]  

(A.4)

The first term of the right-hand-side of Eq. A.4 is the local acceleration (the independent coordinate variables, \((x, y, z)\), are all kept constant), and the second term is the convective acceleration due to the different velocities of the fluid particles over different positions.

**A2. Gauss convergence theorem**

Gauss divergence theorem is

\[
\iiint_S \mathbf{T} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{T} dV
\]  

(A.5)

**A3. Transport theorem**

The transport theorem is a very important tool to derive the equation for fluid motion. For a given physical function, \( f \), in the fluid domain, \( V \), the transport theorem gives

\[
\frac{D}{Dt} \iiint_V f dV = \iiint_V \left[ \frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{V}) \right] dV
\]  

(A.6)
The details of the derivation of the transport theorem can be found in [35] (pages 57-59).

References